# Homework 3 Solutions

1. Let $P_{ij}$ be the permutation matrix that represents the row operation "Swap Row $i$ and Row $j$". Use the Gauss-Jordan algorithm to show that $P_{ij}^{-1} = P_{ij}$.

   To reduce $P_{ij}$ to RREF, we simply need to swap row $i$ and row $j$. Applying this row operation to the augmented matrix yields:

   $$\left[\begin{array}{c|c} P_{ij} & I \end{array}\right] \xrightarrow{\text{Swap Ri and Rj}} \left[\begin{array}{c|c} I & P_{ij} \end{array}\right]$$

   It follows that $P_{ij}^{-1} = P_{ij}$.

2. In class, we discussed elimination and permutation matrices. Multiplication on the left by an elimination matrix corresponds to a row replacement and multiplication on the left by a permutation matrix corresponds to swapping two rows. These are two of the three row operations needed to reduce a matrix to RREF. The final row operation is scaling.

   (a) Write down the $3 \times 3$ matrix $S_2$ that corresponds to multiplying the second row of a matrix by $x$ (where $x \neq 0$). Verify your answer is correct by computing $S_2A$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

   Let $S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

   $$S_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4x & 5x & 6x \\ 7 & 8 & 9 \end{bmatrix}$$

   (b) Describe the $n \times n$ matrix $S_i$ that corresponds to multiplying the $i$th row of a matrix by $x$ (where $x \neq 0$).

   $S_i$ is simply the identity matrix with the $i$th diagonal entry replaced by $x$.

   (c) Elimination matrices and permutation matrices are invertible. Is $S_i$ invertible? If so, describe its inverse.

   $S_i^{-1}$ is the matrix obtained from the identity by changing the $i$th entry to $\frac{1}{x}$. In multiplying $S_iS_i^{-1}$, the $i$th entry on the diagonal becomes $x\frac{1}{x} = 1$.

3. Determine which of the following sets are subspaces. If a set is a subspace, prove it. If a set is not a subspace, explain why.

(a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; xy = 0 \right\} \subset \mathbb{R}^2$.

Since $1 \cdot 0 = 0$, we have that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$. However, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V$ because $1 \cdot 1 \neq 0$. Thus $V$ is not a subspace.

(b) $W = \left\{ \begin{bmatrix} 2x - y \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \subset \mathbb{R}^2$

Method 1:

- When $x = y = 0$, we have $\begin{bmatrix} 2x - y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$;
- Let $\begin{bmatrix} 2x_1 - y_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} 2x_2 - y_2 \\ y_2 \end{bmatrix} \in W$. Then

$$\begin{bmatrix} 2x_1 - y_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 2x_2 - y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1 + x_2) - (y_1 + y_2) \\ y_1 + y_2 \end{bmatrix} \in W$$

- Let $\begin{bmatrix} 2x - y \\ y \end{bmatrix} \in W$ and $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} 2x - y \\ y \end{bmatrix} = \begin{bmatrix} 2cx - cy \\ cy \end{bmatrix} \in W$$

Since $cx, cy \in \mathbb{R}$.

Method 2:

Since $\begin{bmatrix} 2x - y \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, any vector in $W$ can be expressed as a linear combination of the vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Moreover, since $x$ and $y$ can be any real numbers, $W$ is simply the span of these two vectors. Since spans are always subspaces, $W$ is a subspace.

(c) $U = \left\{ \begin{bmatrix} x + y + z \\ 2x - y \\ 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathbb{R}^3$

Since the final component of $\begin{bmatrix} x + y + z \\ 2x - y \\ 1 \end{bmatrix}$ is never zero, have that $\mathbf{0} \notin U$. Hence $U$ is not a subspace.

4. Let's consider intersections and unions of subspaces.

   (a)
   - Let $L_1$ and $L_2$ be two subspaces of $\mathbb{R}^2$ represented by lines. What does $L_1 \cap L_2$ look like?

     If $L_1 \neq L_2$, then $L_1 \cap L_2$ is a point (namely, the origin). If $L_1 = L_2$, then $L_1 \cap L_2$ is a line.

   - Let $L$ and $P$ be subspaces of $\mathbb{R}^3$ represented by a line and a plane, respectively. What does $L \cap P$ look like?

     If $L$ is not in $P$, then $L \cap P$ is a point (namely, the origin). If $L$ is in $P$, then $L \cap P = L$.

   - Let $P_1$ and $P_2$ be subspaces of $\mathbb{R}^3$ represented by planes. What does $P_1 \cap P_2$ look like?

     If $P_1 \neq P_2$, then $P_1 \cap P_2$ looks like a line passing through the origin. If $P_1 = P_2$, then $P_1 \cap P_2$ is a plane.

   (b) Let $V$ and $W$ be subspaces of $\mathbb{R}^n$. Prove that $V \cap W$ is also a subspace.

   - Since $V$ and $W$ are subspaces, $\mathbf{0} \in V$ and $\mathbf{0} \in W$. Hence $\mathbf{0} \in V \cap W$.
   - Let $\mathbf{a}, \mathbf{b} \in V \cap W$. Then $\mathbf{a}, \mathbf{b} \in V$ and $\mathbf{a}, \mathbf{b} \in W$. Since $V$ and $W$ are subspaces, $\mathbf{a} + \mathbf{b} \in V$ and $\mathbf{a} + \mathbf{b} \in W$. It follows that $\mathbf{a} + \mathbf{b} \in V \cap W$.
   - Let $\mathbf{a} \in V \cap W$ and $c \in \mathbb{R}$. Then $\mathbf{a} \in V$ and $\mathbf{a} \in W$. Since $V$ and $W$ are subspaces, $c\mathbf{a} \in V$ and $c\mathbf{a} \in W$. Hence $c\mathbf{a} \in V \cap W$.

   (c) Let $V$ and $W$ be subspaces of $\mathbb{R}^n$. Show that $V \cup W$ is not necessarily a subspace by providing a counterexample.

   Let $V = \text{Span}\{\begin{bmatrix}1\\0\end{bmatrix}\}$ and $W = \text{Span}\{\begin{bmatrix}0\\1\end{bmatrix}\}$ in $\mathbb{R}^2$. Then $V$ is represented by the $x-$axis and $W$ is represented by the $y-$axis. Hence $V \cup W$ is the union of the two axes. Now $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in V \cup W$. However, $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin V \cup W$. thus $V \cup W$ is not a subspace.

5. True/False. If a statement is true, explain why. If a statement is false, explain why or provide a counterexample.

(a) $N(A) = N(A^T)$.

False. Suppose $A$ is an $2 \times 3$ matrix. Then $N(A)$ is a subspace of $\mathbb{R}^3$ and so it is made of 3-dimensional vectors. $N(A^T)$, on the other hand is a subspace of $\mathbb{R}^2$ (since $A^T$ is a $3 \times 2$ matrix) and so it is made of 2-dimensional vectors. Thus $N(A) \neq N(A^T)$.

(b) The column space of $\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$ is a plane in $\mathbb{R}^3$.

False. Notice that the second column of the matrix is a multiple of the first column

$$\begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Thus any linear combination of these two columns is of the form

$$a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b(-2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}) = (a - 2b) \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

and is hence a multiple of the first column. Thus the column space is the set of scalar multiples of the first column, which is a line in $\mathbb{R}^3$, not a plane.

We can also see this geometrically, the two columns lie on the same line in $\mathbb{R}^3$. Hence the span is the line, not a plane.

6. Explain why there does not exist a matrix whose column space is a plane in $\mathbb{R}^3$ and whose nullspace is a line in $\mathbb{R}^4$.

Suppose there is such a matrix $A$. Then $A$ is a $3 \times 4$ matrix, since $C(A)$ is a subspace of $\mathbb{R}^3$ and $N(A)$ is a subspace of $\mathbb{R}^4$. Since $N(A)$ is a line, it is spanned by a single vector. This means that the system $A\mathbf{x} = \mathbf{0}$ has a single free variable, and hence $A$, which has four columns, must have three pivot columns. Since $A$ has 3 rows, it has a pivot in every single row. By problem 3a on Worksheet9.17, this implies that $C(A) = \mathbb{R}^3$. But, we know that $C(A)$ is a plane, not all of $\mathbb{R}^3$. Hence there does not exist such a matrix.

7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$.

(a) Use Julia to reduce $A$ to RREF. Take a screenshot of your code and add it as its own page.

RREF: $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) What are the pivot variables and free variables of the system $A\mathbf{x} = \mathbf{0}$?

The pivot variables are $x_1$ and $x_2$; the free variables are $x_3, x_4$, and $x_5$.

(c) Find the general solution to $A\mathbf{x} = \mathbf{0}$ and write it as a linear combination of special solutions.
The underlying system of equations is

$$\begin{aligned} x_1 - x_3 - 2x_4 - 3x_5 &= 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 &= 0 \end{aligned}$$

which we can rewrite in terms of the free variables:

$$\begin{aligned} x_1 &= x_3 + 2x_4 + 3x_5 \\ x_2 &= -2x_3 - 3x_4 - 4x_5 \end{aligned}$$

Hence the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $x_3, x_4, x_5 \in \mathbb{R}$.

(d) Describe the nullspace of $A$ as a span of vectors. What does the nullspace look like geometrically? In which vector space does it live?

It follows from part (c) that $N(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
Since the vectors are 5-dimensional, $N(A)$ is a subspace of $\mathbb{R}^5$. Since it is the span of three vectors, it looks like 3-dimensional space (inside $\mathbb{R}^5$)

(e) Show that $\mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$. Now use part (c) to find the general solution to $A\mathbf{x} = \mathbf{b}$.

Since $A\mathbf{x}_p = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \mathbf{b}$, we have that $\mathbf{x}_p$ is a solution. Now the general solution is of the form $\mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n \in N(A)$. Hence the general solution is

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 - 1 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 + 1 \\ x_4 \\ x_5 \end{bmatrix}, \text{ where } x_2, x_3, x_4 \in \mathbb{R}.$$

8. Let $A$ and $B$ be any two matrices for which $AB$ is defined. Prove that $C(AB)$ is a subset of $C(A)$.

We need to show that any vector in $C(AB)$ is also a vector in $C(A)$. Let's start with the columns of $AB$. Let $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$. Then $AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_n \end{bmatrix}$. By definition each column $A\mathbf{b}_i$ is a linear combination of the columns of $A$ (with weights coming from $\mathbf{b}_i$). That is, $A\mathbf{b}_i \in C(A)$ for all $i$.

Now let $\mathbf{v} \in C(AB)$. Then $\mathbf{v} = x_1(A\mathbf{b}_1) + \cdots + x_n(A\mathbf{b}_n)$ for some $x_1, \ldots, x_n \in \mathbb{R}$ (i.e. $\mathbf{v}$ is a linear combination of the columns of $AB$). Since $C(A)$ is a subspace, it is closed under addition and scalar multiplication. Since $A\mathbf{b}_i \in C(A)$ for all $i$, it now follows from closure that $\mathbf{v} \in C(A)$. Therefore $C(AB) \subset C(A)$.